

Measurement and Particle Statistics in the Szilard Engine

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A Szilard Engine is a hypothetical device which is able to extract work from a single thermal reservoir by measuring the position of particles within the engine. We derive the amount of work that can be extracted from such a device in the low temperature limit. Interestingly, we show this work is determined by the information gain of the initial measurement rather than by the number and type of particles which constitute the working substance. Our work provides another clear connection between information gain and extractable work in thermodynamical processes.

The laws of thermodynamics are well known for their robustness, namely the fact that they survived both physics revolutions of the 20th century. Neither quantum physics nor relativity have modified our confidence in the fact that the overall energy of a closed system should be conserved, while its entropy tends to the maximum value with time. At low temperatures, however, quantum systems obey completely different statistics to classical ones and one could imagine that bosonic and fermionic statistics, while not breaking the laws of thermodynamics, might still allow us to obtain more work from indistinguishable particles than from the same number of distinguishable ones. This has indeed recently been reported in [1].

In this Letter we show that the performance of a work cycle with indistinguishable particle depends on the information one has and is capable of obtaining about them. The type of particles doing the work, be they classical, quantum, distinguishable or indistinguishable, is only of secondary importance. Our work clarifies why the basics of thermodynamics are independent of particle statistics, a fact that makes thermodynamical laws all the more remarkable. Furthermore, it emphasises that the second law ought to most appropriately be phrased in terms of a trade-off between information gained and work done.

In this letter we use the concept of the Szilard engine (SZE) [2, 3] - a hypothetical device which consists of a cylinder, serving as a heat reservoir at temperature T , containing a single gas particle. The engine works by dividing the cylinder into two halves by an impenetrable barrier and measuring the position of the gas particle. Following the measurement, work can be extracted from this device by exploiting the pressure created by the particle on the barrier. The amount of work is given as $kT \ln(2)$ with k being the Boltzmann constant. As was identified by Bennett [4], the validity of the second thermodynamical law is saved by the fact that the information about the position of the particle before the work has been done, or, equivalently, about the position of the barrier after the work has been done, has to be stored somewhere and erased by "resetting" the engine,

i.e. removing the barrier and moving it to the middle again.

A natural extension of the original SZE is to include more particles into the engine. In e.g. [4, 6, 7] it was shown that correlations have a positive impact on the amount of work that can be extracted from the engine, quantum correlations having a particularly strong impact. A related important type of multi-particle effect is of course particle statistics associated with identical particles. In a recent paper Kim *et. al.* [1] performed a full quantum analysis of a SZE with more than one particle, bringing the role of particle statistics into focus.

In this Letter we clarify the role of particle statistics in a quantum SZE by placing a particular emphasis on the initial measurement. We consider the low temperature limit as we are interested in quantum effects and these are most strongly manifested in this regime. We show that the extractable work is directly connected with the measurement performed on the system. Crucially, for a measurement with M outcomes, the work is bounded from above by $W = kT \ln(M)$ and reaches this bound only for measurements with equiprobable outcomes. In other words, we find that the extractable work is determined by the information gained during the initial measurement, regardless of whether the working medium consists of distinguishable, bosonic or fermionic particles. By performing a detailed analysis we show that the work costs of the different steps do depend on the working medium but conspire to remove this dependence when combined.

Szilard Engine.— Consider N particles (bosons, fermions or distinguishable particles) placed in a cylinder kept at a constant temperature T via the ongoing interaction between the particles and the walls of the cylinder. This interaction is considered to be fast in the sense that the time scale of thermalization is much smaller than any other time scale used. In the first part of the Letter we will also assume the energy levels of the particles in the cylinder, as well as in its parts after inserting the barrier (piston), to be non-degenerate. However this does not exclude the energy levels on the respective side of the piston having the same energies.

The original quantum SZE is used to extract work from a temperature reservoir via the following steps:

1. A piston (modelled by a sufficiently narrow and high potential barrier) is inserted into the cylinder to separate it into two disconnected regions, preventing any tunneling within the relevant time-scales. The work cost of this step shall be called $W1$.

2. A measurement is performed to obtain the number of particles on one side of the cylinder. This measurement might be full, i.e. providing the exact number of particles (or position in the case of distinguishable particles) or partial. The state of particles in the cylinder, after the wall was inserted and before the measurement is performed, is a mixture of possible states obtained via the measurement rather than their coherent superposition. This is due to the fact that interactions with the walls of the cylinder will not only fix the temperature of the gas during the process but also perform a measurement in the number basis in each part of the cylinder, once the barrier is introduced into the cylinder.

3. Depending on the result of the measurement the piston will be allowed to move quasi-statically to a position where the side-ways force acting on the piston will be zero.

4. The piston is then quasi-statically removed by decreasing the strength of the potential.

In the *classical* case there is no need to invest work into the engine except during the erasure procedure - insertion of the piston, its removal as well as the measurement are considered to be "for free". The only stage of extraction of energy is the movement of the piston resulting from the unequal pressure on either side. In contrast, in the *quantum* case one inevitably has to invest energy to create the barrier (step 1.); this can however be recovered during the movement and removal of the barrier (steps 3 and 4). In this picture the stages of movement and removal of the barrier need not anymore be considered as independent actions. The extraction of work can be equally well done by exploiting the force towards the side of the barrier as well as acting on the top of the barrier. Therefore throughout the manuscript we will only work with three phases of the engine - insertion, measurement and movement/removal phase. The combined work-cost of steps 3. and 4. shall be called $W2$.

These four processes represent a closed cycle up to the point where the result of the measurement in step 2 is still stored somewhere. For a full restoration of the original setting one has to erase this information [4].

Low temperature limit.—In the first part of this Letter we will consider the system to be non-degenerate and assume the low temperature limit. Here, the quantum effects of the engine are expected to manifest themselves in the strongest way - for high temperatures all particles became essentially distinguishable as each can be labelled by its energy. In what follows we shall denote the energy levels of the particles in the cylinder

without the barrier as E_i . We consider the condition $kT \ll \Delta E$, where ΔE represents the difference between any two energy levels that could be potentially occupied - for bosons and distinguishable particles it is just the difference between two lowest energy levels of the system and for fermions ΔE represents the difference between the Fermi energy and the nearest higher energy level. Under these assumptions the partition function of any system consisting of N bosons or distinguishable particles will be $Z_b = \exp(-NE_1/kT)$ with E_1 being the lowest energy of the system. For fermions in a non-degenerate system the partition function will have the form $Z_f = \exp\left(-\sum_{i=1}^N E_i/kT\right)$.

We will now calculate the work that can be extracted from the closed cycle of the SZE. This can be defined as $W = kT \ln \frac{Z_B}{Z_A}$ with Z_B being the partition function of the final state and Z_A being the partition function of the initial state.

Insertion of the barrier.—For *bosons* there are two distinct possibilities for this step. Let us denote the energy levels on the left(right) of the barrier as $E_i^{l(R)}$. The first one corresponds to the case where we do not insert the barrier in the middle of the system $E_1^l > E_1^r$. Quantities for this case will be labeled by index n so the final partition function will be $Z_b^n = \exp(-NE_1^r/kT)$, which physically means that all particles condensed in the lower potential well. This will result in a final work $W1_b^n = N(E_1 - E_1^r)$. Irrespective of the exact form of the potential defining the cylinder one can expect $E_1^r > E_1$, as the "living space" of the particles has decreased and thus the work performed is negative. The second possibility corresponds to the case when the barrier is introduced in the middle of the system i.e. $E_1^l = E_1^r$. In this case quantities will be indexed by d and the precision is expected to be such that $E_1^l - E_1^r \ll kT$ holds. Under such an assumption the partition sum will be $Z_b^d = (N+1) \exp(-NE_1^r/kT)$ and the work $W1_b^d = kT \ln(N+1) + W1_b^n$.

For *distinguishable particles* the situation is quite similar to the above bosonic case. Again there is no limit on the number of particles occupying the lowest energy level. The only difference is for the degenerate case, where the particles can choose their positions without change of the total energy of the system. Here the number of possible configurations will be 2^N and the partition function will be $Z_d^d = 2^N \exp(-NE_1^r/kT)$. Here the extractable work $W1_d^d = N(kT \ln 2 + E_1 - E_1^r)$, simply corresponding to the case of joining N independent SZEs.

For *fermions* the situation is distinctly different. We define j as the level for which following equality holds:

$$E_{N-j}^r \leq E_j^l \leq E_{N-j+1}^r. \quad (1)$$

First let us examine the case where there is sharp inequality on the right hand side of (1). The final partition function will be $Z_f^n = \exp\left(-\left(\sum_{i=1}^j E_i^l + \sum_{i=1}^{N-j} E_i^r\right)/kT\right)$

and the extractable work will be $W1_f^n = \sum_{i=1}^N E_i - \left(\sum_{i=1}^j E_i^l + \sum_{i=1}^{N-j} E_i^r \right)$. In contrast, for equality on the right hand side of (1) we get a degeneracy in the energy levels of the whole system when the fermion at the Fermi level can freely choose either side of the cylinder and the partition function becomes $Z_f^d = 2 \exp \left(- \left(\sum_{i=1}^j E_i^l + \sum_{i=1}^{N-j} E_i^r \right) / kT \right)$ and the extractable work will be $W1_f^d = kT \ln(2) + W1_f^n$. It is worth to mention that in contrast to the bosonic case there are potentially many (in the order of N) possibilities for choosing the position of the barrier to reach the equalized position, but insertion into the middle of the container will cost less work only for N odd.

Measurement.— In the second step a measurement on the system is performed. This is only non-trivial if the energy levels of the final system (after inserting the barrier) are degenerate (with partition functions and works labelled by d) as in the other case it is just a single-outcome measurement confirming that all bosons/distinguishable particles are in the deeper well (larger part of the piston) or the fermions are distributed within the piston in a way expectable by the distribution of energy levels.

In the non-trivial bosonic case, the full measurement will have $N + 1$ possible outcomes counting the number of particles on the left hand side. In the fermionic case the measurement will be binary and for distinguishable particles the number of possible outcomes will be 2^N (specifying the position of every single particle).

Movement and removal of piston.— In the *classical* case one would extract the work by simply moving the piston to its equilibrium position. All extractable work would be extracted within this phase for an infinitely narrow piston. On the other hand, if the piston was removed from a different position than the equilibrium one, part of the potential work would just dissipate due to mixing of gases with different pressures. In the *quantum* case the situation is much more subtle. Here the extraction of work is not straightforward to define physically in connection with possible storages of energy such as excited states of atoms. However, if we stick to the standard definition of the generalized force as $F = \frac{\partial E}{\partial \lambda}$ with λ being a parameter of the barrier (e.g. its position during the movement phase or height during the removal phase) and accept that any such force can be utilized to perform work, we can calculate the extractable work from the partition functions without making specific assumptions on the process itself.

For both bosons and distinguishable particles the initial partition sum of the system is $Z_i^A = \exp(-NE_1^r/kT)$ and final is Z_b . Therefore we get as the extractable work $W2_b = N(E_1^r - E_1)$. For fermions the partition sum is $Z_f^A = \exp \left(- \left(\sum_{i=1}^j E_i^l + \sum_{i=1}^{N-j} E_i^r \right) / kT \right)$ and the final one Z_f ; the resulting work is $W2_f = \sum_{i=1}^j E_i^l +$

$$\sum_{i=1}^{N-j} E_i^r - \sum_{i=1}^N E_i.$$

Summing up.—The total work gained (or paid) during the relevant parts of the cycle is then $W_b = W_d = W_f = 0$ for non-degenerate cases with trivial measurements. The degenerate cases are much more interesting. In the case of distinguishable particles $W_d = NkT \ln(2)$, corresponding to N independent SZE's, for bosons we get $W_b = kT \ln(N + 1)$ and for fermions $W_f = kT \ln(2)$. All these results (even for non-degenerate cases) are connected by a unifying formula of the form

$$W = kT \ln(M) \quad (2)$$

with M being the number of possible measurement outcomes, each of them occurring with equal probability. The work corresponds to the energy needed to erase a memory able to store the result of the measurement.

We note that S. W. Kim et. al. [1] calculate the work yield during the removal of the barrier differently. They firstly lower the barrier to the point where tunnelling is practically unrestricted. They view any energy that could potentially be extracted here as lost. In the second phase where the barrier is removed and the "living space" of the particles is enlarged they do however extract work. With this approach the net energy gained from the system in the low temperature limit is however always negative if the number of particles $N > 2$. Work could only be gained if all particles are found on one of the sides of the cylinder, resulting in extractable work in the order of kT with probability diminishing with large N . On the other hand, all other measurement results would lead to a loss of energy in the order of ΔE via tunnelling during the removal of the barrier phase.

Coarse-grained measurements.— One may consider another way of exploiting particle statistics to try to violate the second law, by doing a coarse-grained measurement. Let us define a coarse-grained measurement with M outcomes labelled by m (running from 1 to M), each occurring with probability p_m . As $W1$ does not depend on the measurement it will not change. The partition sum for bosons and distinguishable particles after the measurement with outcome m will have the form $Z_m^A = \frac{1}{p_m} \exp(-NE_1^r/kT)$. After the removal phase the partition sum will be $Z_m^B = \exp(-NE_1/kT)$. The extractable work will thus be

$$W = -kT \sum_{m=1}^M p_m \ln(p_m), \quad (3)$$

which holds also for fermions with $M = 2$ and $p_1 = p_2 = 1/2$. From this we see that also in this case the extractable work does not depend on the actual number and type of particles in the system, but only on the measurement and its possible outcomes (taking into account the fact that possible measurements are limited by the number and type of particles in the system). For a fixed

number of outcomes M the extractable work is maximized for a measurement with equal probability of each outcome and will gain $W_{\max} = kT \ln(M)$.

We note that the result in Eq. (3) can be rewritten as $W = TS$ with

$$S = -k \sum_{m=1}^M p_m \ln(p_m), \quad (4)$$

being the entropy of the measurement outcomes. This definition exactly corresponds to the definition of Gibbs entropy except that here one deals with the probabilities of measurement outcomes rather than with the probabilities of microstates. One might view this correspondence in the following way: by removing entropy S from the system by a measurement one is able to extract exactly $W = TS$ of work from the system before it turns back to its original state. The extracted entropy has to be stored somewhere, or erased, costing exactly the same amount of work again [4]. (A net work yield can nevertheless be obtained if two reservoirs with different temperatures exist and the erasure is performed whilst in contact with the colder one.)

For a measurement of distinguishable particles, where one only measures the number of particles on each side, the extractable work will be

$$W_d = -kT \sum_{m=0}^N 2^{-N} \binom{N}{m} \ln \left(2^{-N} \binom{N}{m} \right), \quad (5)$$

which is smaller than W_b for bosons, where the same measurement represents a full measurement of the system (reaching $\frac{1}{2}W_b$ for the limit of large N [8]). This can be explained by the smaller information value of the measurement outcome (with the same number of possible outcomes $N+1$) for distinguishable particles in comparison to bosons - whereas for bosons any result is equally probable, for distinguishable particles only the results close to balanced are likely.

Degeneracy.—Let us briefly discuss the possible degeneracies of energy levels in the system. For bosons and distinguishable particles they only play a role for the lowest energy level. If the degeneracy can be revealed by the measurement, the system will correspond to an advanced SZE with more than one barrier and more complicated measurement. If the degeneracy cannot be revealed, influence will be exactly cancelled for the extractable work, however it would increase the work needed to insert the barrier.

For fermions, if the Fermi energy level of the system is degenerate, more than one fermion can occupy the same energy. This can increase the number of particles actually performing work in the system up to N . In this scenario fermions behave exactly like bosons, condensing into the lowest energy level and performing the same amount of work as bosons (if the degeneracies cannot be

revealed by measurement) or distinguishable particles (if degeneracies can be revealed by the measurement).

Summary and discussion.—We performed a detailed analysis of a Szilard engine in the low temperature limit where the working medium is either distinguishable particles, bosons or fermions. We showed that the extractable work is determined by the information gain of the measurement performed on the system, regardless of the working medium. We demonstrated in detail how things conspire to remove the dependence on the working medium in that sense. This latter contribution is arguably of the same type as Bennett's exorcism of Maxwell's daemon [4] in that it shows in what exact way the second law is not violated in this process.

We also showed that if a full measurement on the system is performed (which is associated with a different amount of information gain in the different cases), distinguishable particles exhibit a much larger potential to deliver work (scaling linearly with the number of particles) relative to bosons (where it scales logarithmically). Fermions can only provide a fixed amount of work independent of the number of particles.

Moreover we showed that for coarse-grained measurements the extractable work is again determined by the information gain of the measurement. This clarifies why the same kind of measurement (measuring the number of particles on either side of the piston) extracts more energy for bosons than for distinguishable particles.

It would be interesting to provide experimental evidence for the results obtained, especially for the possibility to extract work with balanced measurement outcomes. One could think about cold atoms kept at a stable temperature by a (larger amount) of different atoms, as suggested in [1]. Another option would be to use photons in micro-cavities as the working media. Here the confinement potential is easily controllable and the barrier could consist just from an inserted mirror.

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